Numerical Algorithm for One and Two-Step Hybrid Block Methods for the Solution of First Order Initial Value Problems in Ordinary Differential Equations

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Abstract: In this paper, we developed a numerical Algorithm for one and two-step hybrid block methods for the numerical solution of first order initial value problems in ordinary differential equations using method of collocation and interpolation of Taylor's series as approximate solution to give a system of non linear equations which was solved to give a hybrid block method. To further justify the usability and effectiveness of this new hybrid block method, the basic properties of the hybrid block scheme was investigated and found to be zero-stable, consistent and convergent. The derived scheme was tested on some numerical examples and was found to give better approximation than the existing methods. The errors displayed after solving some selected initial value problems, revealed that, it is better to increase L (Derivative) rather than the step length k as shown in our numerical results. Also, It was difficult to satisfy the zero-stability for larger k. In addition, the new method converges faster with lesser time of computation, which address the setback associated with other methods in the literature. Finally, the new method has order of accuracy for one-step as order Ten while order Eighteen for two-step.

Keywords: Block Method, Initial Value Problems, Hybrid, One-Step, Two-Step, Taylor Series

1. Introduction

Mathematical models are developed to help in understanding physical phenomena. These models often yield equations that contains some derivatives of an unknown function of one or several variables. Such equations are called Differential Equation (DE) [1]. Our focus in this research work will be on first order ordinary differential equations of the form:

$$y^{(1)} = f(x, y) \ , \ y(x) = y_0 \ , \ x \in [a, b] \qquad (1)$$

where f is continuous in $[a, b]$, such equation in (1) is often encountered in areas such as control theory, chemical kinetics, circuit theory, biological sciences and many others. The fact that most often, this class of equations cannot be solved analytically, then the approach in developing several numerical methods to approximate the solution of problem (1) comes in [9, 12]. Some approaches to this alternative method (i.e. numerical method) include the Nystrom type methods, the self-starting Runge-Kutta type methods which involve several function evaluations per step, and the linear multi-step methods, which requires fewer function evaluations per step [2]. Conventionally, linear multi-step methods are implemented in the predictor-corrector mode which is prone to error propagation as the integration process progresses [10]. The disadvantages associated with the predictor-corrector method led to the development of block methods from linear multi-step methods [10]. Apart from being self-starting, the method does not require the development of predictors separately, and evaluates fewer functions per step when compared to the Runge-Kutta type methods [13]. In this paper, we are motivated by the work of Zurni Omar & Oluwaseun adeyeye to develop two-step block method for solving first order ordinary differential equations using collocation and interpolation approach [8]. For the quest for better accuracy, we develop one and two-step hybrid block methods using same approach, this hybrid method has great properties, easy to use and gives a better performance. Recent work on hybrid

block methods can be seen in [1, 4, 5, 7, 10, 15, 16, 17]. We note that all of these methods are governed by the Dahlquist's barrier conditions [3]. However, this barrier conditions have been circumvented in [2, 4, 5, 14]. This work is motivated by the need to develop a new numerical method which can handle system of equations of initial value problems of first order ordinary differential equations. The new method is expected to converge faster with lesser time of computation. Hence address setback associated with other methods in the literature. The method has order of accuracy for one-step as P=10 while two-step P=18, the accessibly and accuracy of the methods shall be illustrated on the table.

2. Development of Hybrid Block Methods

In this section, we intend to derive one and two-step hybrid block methods which will be use to generate the main method and other methods required to set up the hybrid block methods. We set out by approximating the analytical solution of problem (1) with a Taylor's series. According to [6], the approximate solution of equation (1) *for* $k = 1, l = 2$ is given as:

$$\sum_{j=0}^{k-1} \alpha_j y_{n+j} + \sum_{i=1}^{l} h^i \sum_{j=0}^{k} \beta_{ij} y'_{n+j} \ ; \ \alpha_k = +1 \qquad (2)$$

Similarly, the two-step hybrid block method for $k = 2, l = 2$ as:

$$\sum_{j=0}^{k-2} \alpha_j y_{n+j} + \sum_{i=1}^{l} h^i \sum_{j=0}^{k} \beta_{ij} y'_{n+j} \ ; \ \alpha_k = +1 \qquad (3)$$

In order to obtain the main method for one-step we rewrite equation (2) to be:

$$y_{n+1} = \alpha_0 y_n + h\left[\beta_{10} f_n + \beta_{1\frac{1}{4}} f_{n+\frac{1}{4}} + \beta_{1\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_{1\frac{3}{4}} f_{n+\frac{3}{4}} + \beta_{11} f_{n+1}\right] + h^2\left[\beta_{20} g_n + \beta_{2\frac{1}{4}} g_{n+\frac{1}{4}} + \beta_{2\frac{1}{2}} g_{n+\frac{1}{2}} + \beta_{2\frac{3}{4}} g_{n+\frac{3}{4}} + \beta_{21} g_{n+1}\right] \quad (4)$$

Where $y^{(1)}_{n+j} = f_{n+j} \ and \ y^{(2)}_{n+j} = g_{n+j}$

Expanding (4) by Taylor's series about x_n of each term and substituting back into equation (4), the matrix form Ax=B which was obtained by the expression $y_{n+a} = y(x_n + ah) = y(x_n) + ahy'(x_n) + \frac{(ah)^2}{2!}y''(x_n) + \frac{(ah)^3}{3!}y'''(x_n) + \cdots$ and the coefficients of $h^i y^{(i)}(x_n)$ are equated to give:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & \frac{\left(\frac{1}{4}\right)^2}{2!} & \frac{\left(\frac{1}{2}\right)^2}{2!} & \frac{\left(\frac{3}{4}\right)^2}{2!} & \frac{1}{2!} & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\
0 & 0 & \frac{\left(\frac{1}{4}\right)^3}{3!} & \frac{\left(\frac{1}{2}\right)^3}{3!} & \frac{\left(\frac{3}{4}\right)^3}{3!} & \frac{1}{3!} & 0 & \frac{\left(\frac{1}{4}\right)^2}{2!} & \frac{\left(\frac{1}{2}\right)^2}{2!} & \frac{\left(\frac{3}{4}\right)^2}{2!} & \frac{1}{2!} \\
0 & 0 & \frac{\left(\frac{1}{4}\right)^4}{4!} & \frac{\left(\frac{1}{2}\right)^4}{4!} & \frac{\left(\frac{3}{4}\right)^4}{4!} & \frac{1}{4!} & 0 & \frac{\left(\frac{1}{4}\right)^3}{3!} & \frac{\left(\frac{1}{2}\right)^3}{3!} & \frac{\left(\frac{3}{4}\right)^3}{3!} & \frac{1}{3!} \\
0 & 0 & \frac{\left(\frac{1}{4}\right)^5}{5!} & \frac{\left(\frac{1}{2}\right)^5}{5!} & \frac{\left(\frac{3}{4}\right)^5}{5!} & \frac{1}{5!} & 0 & \frac{\left(\frac{1}{4}\right)^4}{4!} & \frac{\left(\frac{1}{2}\right)^4}{4!} & \frac{\left(\frac{3}{4}\right)^4}{4!} & \frac{1}{4!} \\
0 & 0 & \frac{\left(\frac{1}{4}\right)^6}{6!} & \frac{\left(\frac{1}{2}\right)^6}{6!} & \frac{\left(\frac{3}{4}\right)^6}{6!} & \frac{1}{6!} & 0 & \frac{\left(\frac{1}{4}\right)^5}{5!} & \frac{\left(\frac{1}{2}\right)^5}{5!} & \frac{\left(\frac{3}{4}\right)^5}{5!} & \frac{1}{5!} \\
0 & 0 & \frac{\left(\frac{1}{4}\right)^7}{7!} & \frac{\left(\frac{1}{2}\right)^7}{7!} & \frac{\left(\frac{3}{4}\right)^7}{7!} & \frac{1}{7!} & 0 & \frac{\left(\frac{1}{4}\right)^6}{6!} & \frac{\left(\frac{1}{2}\right)^6}{6!} & \frac{\left(\frac{3}{4}\right)^6}{6!} & \frac{1}{6!} \\
0 & 0 & \frac{\left(\frac{1}{4}\right)^8}{8!} & \frac{\left(\frac{1}{2}\right)^8}{8!} & \frac{\left(\frac{3}{4}\right)^8}{8!} & \frac{1}{8!} & 0 & \frac{\left(\frac{1}{4}\right)^7}{7!} & \frac{\left(\frac{1}{2}\right)^7}{7!} & \frac{\left(\frac{3}{4}\right)^7}{7!} & \frac{1}{7!} \\
0 & 0 & \frac{\left(\frac{1}{4}\right)^9}{9!} & \frac{\left(\frac{1}{2}\right)^9}{9!} & \frac{\left(\frac{3}{4}\right)^9}{9!} & \frac{1}{9!} & 0 & \frac{\left(\frac{1}{4}\right)^8}{8!} & \frac{\left(\frac{1}{2}\right)^8}{8!} & \frac{\left(\frac{3}{4}\right)^8}{8!} & \frac{1}{8!}
\end{bmatrix}
\begin{bmatrix}
\alpha_0 \\ \beta_{10} \\ \beta_{1\frac{1}{4}} \\ \beta_{1\frac{1}{2}} \\ \beta_{1\frac{3}{4}} \\ \beta_{11} \\ \beta_{20} \\ \beta_{2\frac{1}{4}} \\ \beta_{2\frac{1}{2}} \\ \beta_{2\frac{3}{4}} \\ \beta_{21}
\end{bmatrix}
=
\begin{bmatrix}
1 \\ 1 \\ \frac{1}{2!} \\ \frac{1}{3!} \\ \frac{1}{4!} \\ \frac{1}{5!} \\ \frac{1}{6!} \\ \frac{1}{7!} \\ \frac{1}{8!} \\ \frac{1}{9!} \\ \frac{1}{10!}
\end{bmatrix}
\qquad (5)
$$

Solving equation (5) we have: $\left(\alpha_0, \beta_{10}, \beta_{1\frac{1}{4}}, \beta_{1\frac{1}{2}}, \beta_{1\frac{3}{4}}, \beta_{11}, \beta_{20}, \beta_{2\frac{1}{4}}, \beta_{2\frac{1}{2}}, \beta_{2\frac{3}{4}}, \beta_{21} \right)^T$

$= \left(1, \dfrac{1601}{17010}, \dfrac{2048}{8505}, \dfrac{104}{315}, \dfrac{2048}{8505}, \dfrac{1601}{17010}, \dfrac{29}{11340}, -\dfrac{32}{2835}, 0, \dfrac{32}{2835}, -\dfrac{29}{11340} \right)^T$ Substituting   the   value   of

$\alpha_0, \beta_{10}, \beta_{1\frac{1}{4}}, \beta_{1\frac{1}{2}}, \beta_{1\frac{3}{4}}, \beta_{11}, \beta_{20}, \beta_{2\frac{1}{4}}, \beta_{2\frac{1}{2}}, \beta_{2\frac{3}{4}}, \beta_{21}$ into equation (4) we have

$$y_{n+1} = y_n + \frac{h}{17010}\left[1601f_n + 4096f_{n+\frac{1}{4}} + 5616f_{n+\frac{1}{2}} + 4096f_{n+\frac{3}{4}} + 1601f_{n+1} \right] + \frac{h^2}{11340}\left[29g_n - 128g_{n+\frac{1}{4}} + 128g_{n+\frac{3}{4}} - 29g_{n+1} \right] \quad (6)$$

However, adopting the method presented in Equation (6), we require the introduction of additional hybrid off points in other to implement the method. Hence the need to adopt hybrid block methods will be necessary. Hybrid block methods have been found to give better approximation as seen in [10]. Following the same steps adopted above we have the following additional methods as:

$$y_{n+\frac{1}{4}} = y_n + \frac{h}{17418240}\left[1539551f_n + 1429936f_{n+\frac{1}{4}} + 711936f_{n+\frac{1}{2}} + 1429936f_{n+\frac{3}{4}} + 59681f_{n+1} \right]$$
$$+ \frac{h^2}{11612160}\left[26051g_n - 249656g_{n+\frac{1}{4}} - 183708g_{n+\frac{1}{2}} - 49720g_{n+\frac{3}{4}} - 2237g_{n+1} \right] \quad (7)$$

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{272160}\left[24463f_n + 52928f_{n+\frac{1}{4}} + 44928f_{n+\frac{1}{2}} + 12608f_{n+\frac{3}{4}} + 1153f_{n+1} \right]$$
$$+ \frac{h^2}{181440}\left[421g_n - 3040g_{n+\frac{1}{4}} - 4536g_{n+\frac{1}{2}} - 992g_{n+\frac{3}{4}} - 43g_{n+1} \right] \quad (8)$$

$$y_{n+\frac{3}{4}} = y_n + \frac{h}{71680}\left[6501f_n + 14736f_{n+\frac{1}{4}} + 20736f_{n+\frac{1}{2}} + 11376f_{n+\frac{3}{4}} + 411f_{n+1} \right]$$
$$+ \frac{h^2}{143360}\left[339g_n - 143360g_{n+\frac{1}{4}} - 2268g_{n+\frac{1}{2}} - 1464g_{n+\frac{3}{4}} - 45g_{n+1} \right] \quad (9)$$

Hence, equations (6) to (9) are the required one-step hybrid block method for the solution of equation (1).

Similarly, by expanding equation (3) and after some algebraic simplifications, we obtained the two-step hybrid block method as:

$$y_{n+2} = y_n + \frac{h}{379915160625}\left[\begin{array}{l} 28982995193f_n - 12506560512f_{n+\frac{1}{4}} - 130637631488f_{n+\frac{1}{2}} + 192253458432f_{n+\frac{3}{4}} \\ + 603645798000f_{n+1} + 192253458432f_{n+\frac{5}{4}} - 130637631488f_{n+\frac{3}{2}} \\ - 12506560512f_{n+\frac{7}{4}} - 28982995193f_{n+2} \end{array} \right]$$
$$+ \frac{h^2}{10854718875}\left[\begin{array}{l} 28982995193g_n - 430636032g_{n+\frac{1}{4}} - 1705395328g_{n+\frac{1}{2}} - 2528949248g_{n+\frac{3}{4}} \\ + 2528949248g_{n+\frac{5}{4}} + 1705395328g_{n+\frac{3}{2}} + 430636032g_{n+\frac{7}{4}} - 17427733g_{n+2} \end{array} \right] \quad (10)$$

with additional method as:

$$y_{n+\frac{1}{4}} = y_n + \frac{h}{398369919467520000} \begin{bmatrix} 29781446118518135f_n - 72999059705411552f_{n+\frac{1}{4}} \\ -407731775173649472f_{n+\frac{1}{2}} - 407832499620458976f_{n+\frac{3}{4}} \\ +246530180391936000f_{n+1} + 499167648739062240f_{n+\frac{5}{4}} \\ +190943626263487552f_{n+\frac{3}{2}} + 5134325041819744y'_{f_{n+\frac{7}{4}}} \\ +481342972670601f_{n+2} \end{bmatrix} + \frac{h^2}{11381997699072000} \begin{bmatrix} 17504931897415g_n - 531530319976912g_{n+\frac{1}{4}} \\ -4889514566384g_{n+\frac{1}{2}} - 5635844792319056g_{n+\frac{3}{4}} \\ -6369067687882900g_{n+1} - 3193665436195920g_{n+\frac{5}{4}} \\ -656393465454272g_{n+\frac{3}{2}} - 45493562882512g_{n+\frac{7}{4}} \\ -617613995961g_{n+2} \end{bmatrix} \quad (11)$$

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{389033124480000} \begin{bmatrix} 29177233452383f_n - 35206384310016f_{n+\frac{1}{4}} \\ -372815519345360f_{n+\frac{1}{2}} - 408952708507392f_{n+\frac{3}{4}} \\ +256594483488000f_{n+1} + 509476691764992f_{n+\frac{5}{4}} \\ +194180807928016f_{n+\frac{3}{2}} + 21573850863360f_{n+\frac{7}{4}} \\ +488106906017f_{n+2} \end{bmatrix} + \frac{h^2}{11115232128000} \begin{bmatrix} 17203355203g_n - 488723619456g_{n+\frac{1}{4}} \\ -2470464887800g_{n+\frac{1}{2}} - 5808531516032g_{n+\frac{3}{4}} \\ -6509109707100g_{n+1} - 3251970558592g_{n+\frac{5}{4}} \\ -667007866616g_{n+\frac{3}{2}} - 46168571520g_{n+\frac{7}{4}} \\ -626205757g_{n+2} \end{bmatrix} \quad (12)$$

$$y_{n+\frac{3}{4}} = y_n + \frac{h}{546460794880000} \begin{bmatrix} 40994004828367f_n - 48740915874528f_{n+\frac{1}{4}} \\ -466038920273472f_{n+\frac{1}{2}} - 513696859296992f_{n+\frac{3}{4}} \\ +369189338112000f_{n+1} + 722210413303008f_{n+\frac{5}{4}} \\ +274738316926528f_{n+\frac{3}{2}} + 30500446325472f_{n+\frac{7}{4}} \\ +689772109617f_{n+2} \end{bmatrix} + \frac{h^2}{15613165568000} \begin{bmatrix} 24176900127g_n - 685186467408g_{n+\frac{1}{4}} \\ -3406147722432g_{n+\frac{1}{2}} - 8357755464912g_{n+\frac{3}{4}} \\ -9237040312500g_{n+1} - 4604093435088g_{n+\frac{5}{4}} \\ -943397477568g_{n+\frac{3}{2}} - 65263232592g_{n+\frac{7}{4}} \\ -884881377g_{n+2} \end{bmatrix} \quad (13)$$

$$y_{n+1} = y_n + \frac{h}{6078642570000} \begin{bmatrix} 456032697335f_n - 540514522112f_{n+\frac{1}{4}} \\ -5159193049920f_{n+\frac{1}{2}} - 5159193049920f_{n+\frac{3}{4}} \\ +4829166384000f_{n+1} + 8092995671040f_{n+\frac{5}{4}} \\ +3068990946112f_{n+\frac{3}{2}} + 340409553920f_{n+\frac{7}{4}} \\ +7695225753f_{n+2} \end{bmatrix} + \frac{h^2}{6576570234234000} \begin{bmatrix} 10185173516095g_n - 288488734964224g_{n+\frac{1}{4}} \\ -1432144014406560g_{n+\frac{1}{2}} - 3481713903643136g_{n+\frac{3}{4}} \\ -3946586523279400g_{n+1} - 1949494364858880g_{n+\frac{5}{4}} \\ -398892736240544g_{n+\frac{3}{2}} - 27578420984320g_{n+\frac{7}{4}} \\ -373801932081g_{n+2} \end{bmatrix} \quad (14)$$

$$y_{n+\frac{5}{4}} = y_n + \frac{h}{637391871148032} \begin{bmatrix} 47820846879335f_n - 56558247306720f_{n+\frac{1}{4}} \\ -539628348983360f_{n+\frac{1}{2}} - 519838446190560f_{n+\frac{3}{4}} \\ +582127150080000f_{n+1} + 921723796570080f_{n+\frac{5}{4}} \\ +324414220486720f_{n+\frac{3}{2}} + 35868877563360f_{n+\frac{7}{4}} \\ +809989836185f_{n+2} \end{bmatrix} + \frac{h^2}{91055981592576} \begin{bmatrix} 141033792275g_n - 3993052035600g_{n+\frac{1}{4}} \\ -19807786980800g_{n+\frac{1}{2}} - 48065436806800g_{n+\frac{3}{4}} \\ -53870419102500g_{n+1} - 27528065978000g_{n+\frac{5}{4}} \\ -5558760625600g_{n+\frac{3}{2}} - 383570614800g_{n+\frac{7}{4}} \\ -5194738925g_{n+2} \end{bmatrix} \quad (15)$$

$$y_{n+\frac{3}{2}} = y_n + \frac{h}{533653120000} \begin{bmatrix} 40041810935f_n - 47161274112f_{n+\frac{1}{4}} \\ -449867959632f_{n+\frac{1}{2}} - 428819136256f_{n+\frac{3}{4}} \\ +495938016000f_{n+1} + 831029149440f_{n+\frac{5}{4}} \\ +327904780112f_{n+\frac{3}{2}} + 30726565632f_{n+\frac{7}{4}} \\ +687727881f_{n+2} \end{bmatrix} + \frac{h^2}{15247232000} \begin{bmatrix} 23621115g_n - 668230272g_{n+\frac{1}{4}} \\ -3310470072g_{n+\frac{1}{2}} - 8013188736g_{n+\frac{3}{4}} \\ -8928819900g_{n+1} - 4415483520yg_{n+\frac{5}{4}} \\ -993333432g_{n+\frac{3}{2}} - 65503872g_{n+\frac{7}{4}} \\ -881541g_{n+2} \end{bmatrix} \quad (16)$$

$$y_{n+\frac{7}{4}} = y_n + \frac{h}{1161428336640000} \begin{bmatrix} 87199796486369f_n - 100191396723488f_{n+\frac{1}{4}} \\ -956055718188480f_{n+\frac{1}{2}} - 867566432391456f_{n+\frac{3}{4}} \\ +1126642320384000f_{n+1} + 1776748868948256f_{n+\frac{5}{4}} \\ +789353615447488f_{n+\frac{3}{2}} + 174591779912480f_{n+\frac{7}{4}} \\ +1776755244831f_{n+2} \end{bmatrix} + \frac{h^2}{232285667328000} \begin{bmatrix} 360340582903g_n - 10143840219856g_{n+\frac{1}{4}} \\ -49890409732800g_{n+\frac{1}{2}} - 119295120874832g_{n+\frac{3}{4}} \\ -129980973222100g_{n+\frac{3}{2}} - 60898965421392g_{n+\frac{5}{4}} \\ -10205228686016g_{n+\frac{3}{2}} - 1632157389520g_{n+\frac{7}{4}} \\ -15701442057g_{n+2} \end{bmatrix} \quad (17)$$

Hence, equation (10) to (17) are the required two-step hybrid block method for the solution of equation (1).

3. Basic Properties of the Methods

3.1. Order of One-Step Method

We define a linear operator L by:

$$L\left[y(x):h\right] = \sum_{j=0}^{k}\left[\alpha_j y\left(x_n + jh\right) - h\beta_j y'\left(x_n + jh\right)\right] \tag{18}$$

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a,b]$. Expanding $y(x_n + jh)$ *and* $y'(x_n + jh)$ by Taylor's series about the point x_n and collecting like terms in h and y gives:

$$L\left[y(x):h\right] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + ... + C_p h^p y^p(x) \text{ [10]} \tag{19}$$

Definition 3.1
According to [10], the differential equation (18) and the associated LMM are said to be of order p if

$$C_0 = C_1 = C_2 = ...C_p = 0, C_{p+1} \neq 0 \tag{20}$$

Definition 3.2
The term C_{p+1} is called error constant and it implies that the local truncation error is given by

$$E_{p+1} = C_{p+1} h^{p+1} y^{(p+1)}(x_n) + O\left(h^{p+2}\right) \text{ [10]} \tag{21}$$

Following Definition (3.1) and (3.2), we obtained the order of one-step method as;

$$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0 = C_7 = 0 = C_8 = C_9 = C_{10} = 0, i.e \ P = 10$$

With error constant as $\left(\dfrac{1}{1030045040640}, \dfrac{551}{13184576520192 00}, \dfrac{1}{2060090081280}, \dfrac{1}{1808583884800}\right)^T$

3.2. Consistency and Zero Stability of One-Step Method

To analyze the one-step hybrid block method for zero-stability, the modulus of the roots of its first characteristic polynomial is expected to be simple or less than one [10]. Thus, the corrector of the one-step hybrid block method is normalized to give the first characteristic polynomial [10]. Putting equation (6) to (9) in matrix form as a block we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n-\frac{3}{4}} \\ y_{n-\frac{1}{2}} \\ y_{n-\frac{1}{4}} \\ y_n \end{bmatrix} +$$

$$h\left(\begin{bmatrix} 0 & 0 & 0 & \frac{30791}{3483648} \\ 0 & 0 & 0 & \frac{24463}{272160} \\ 0 & 0 & 0 & \frac{6501}{71680} \\ 0 & 0 & 0 & \frac{1601}{17010} \end{bmatrix}\begin{bmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_n \end{bmatrix} + \begin{bmatrix} \frac{89371}{1088640} & \frac{103}{2520} & \frac{89371}{1088640} & \frac{59681}{17418240} \\ \frac{1654}{8505} & \frac{52}{315} & \frac{394}{8505} & \frac{1153}{272160} \\ \frac{921}{4480} & \frac{81}{280} & \frac{711}{4480} & \frac{411}{71680} \\ \frac{2048}{8505} & \frac{104}{315} & \frac{2048}{8505} & \frac{1601}{17010} \end{bmatrix}\begin{bmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{bmatrix}\right) +$$

$$h^2 \left(\begin{bmatrix} 0 & 0 & 0 & \dfrac{26051}{11612160} \\ 0 & 0 & 0 & \dfrac{421}{181440} \\ 0 & 0 & 0 & \dfrac{339}{143360} \\ 0 & 0 & 0 & \dfrac{29}{11340} \end{bmatrix} \begin{bmatrix} g_{n+\frac{3}{4}} \\ g_{n+\frac{1}{2}} \\ g_{n+\frac{1}{4}} \\ g_n \end{bmatrix} + \begin{bmatrix} -\dfrac{31207}{1451520} & -\dfrac{81}{5120} & -\dfrac{1243}{290304} & -\dfrac{2237}{11612160} \\ -\dfrac{19}{1134} & -\dfrac{1}{40} & \dfrac{31}{5670} & -\dfrac{43}{181440} \\ -1 & -\dfrac{81}{5120} & -\dfrac{183}{17920} & -\dfrac{9}{28672} \\ -\dfrac{32}{2835} & 0 & \dfrac{32}{2835} & -\dfrac{29}{11340} \end{bmatrix} \begin{bmatrix} g_{n+\frac{1}{4}} \\ g_{n+\frac{1}{2}} \\ g_{n+\frac{3}{4}} \\ g_{n+1} \end{bmatrix} \right) \quad (22)$$

The following matrix difference equation will be in the form:

$$A^{(0)}Y_n = A^{(1)}Y_{n-1} + h^{np}\left[B^{(1)}Y_n + B^{(1)}Y_{n-1} \right] \quad (23)$$

The first characteristics polynomial of the matrix in equation (22) is given by

$$R(Z) = \det\left[ZA^0 - A' \right] \quad (24)$$

where

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This implies that;

$$R(Z) = \det\left[Z\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right]$$

$$= \det\left[\begin{bmatrix} Z & 0 & 0 & 0 \\ 0 & Z & 0 & 0 \\ 0 & 0 & Z & 0 \\ 0 & 0 & 0 & Z \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right] = \det\begin{bmatrix} Z & 0 & 0 & -1 \\ 0 & Z & 0 & -1 \\ 0 & 0 & Z & -1 \\ 0 & 0 & 0 & Z-1 \end{bmatrix}$$

$$R(Z) = \left[Z^3(Z-1) \right]$$

If $R(Z) = 0$

$R(Z) = 0$ yields roots $Z = 0, 0, 0, 1$. Therefore, the one-step hybrid block method is zero-stable.

Furthermore, the one-step hybrid block method is said to be consistent if it has order $p \geq 1$ [10]. Therefore, the one-step hybrid block method convergences. Finally, the region of absolute stability is determined by obtaining the stability polynomial. Hence, the stability polynomial for one-step hybrid block method is

gotten as:

$$-\frac{241811}{1371686400}z^4 - \frac{145577}{24494400}z^3 - \frac{4151159}{96446700}z^2 - \frac{1729}{2835} + \frac{3}{4}$$

where $z = \lambda h$. Plotting the roots of the stability polynomial with MATLAB software, displays the region of absolute stability as shown below.
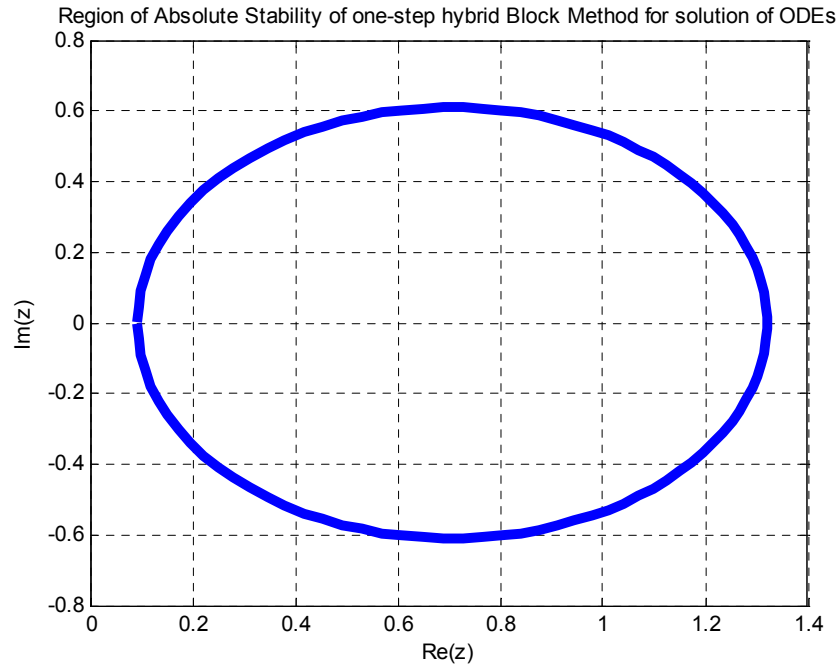
Figure 1. *Region of Absolute stability of one-step hybrid block method.*

3.3. Order of Two-Step Hybrid Block Method

Similarly, we obtained the order of the two-step hybrid block method as:

$$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = C_8 = C_9 = C_{10}$$
$$= C_{11} = C_{12} = C_{13} = C_{14} = C_{15} = C_{16} = C_{17} = C_{18} = 0,$$

i.e $p = 18$

With error constant as

$$\left(\frac{25881301}{203492688559186982731776000}, \frac{408018386767}{666804841870743905015483596800000}, \right.$$
$$\frac{3292740967}{5209412827115186757933465600000}, \frac{193559539}{304894760800523047560806400000},$$
$$\frac{25881301}{40698537711837396546355200000}, \frac{679622015}{106688774699319024802477375488},$$
$$\left. \frac{1523947}{23819903187540863090688000000}, \frac{1282974007}{1944037439856396224534937600000} \right)^T$$

Furthermore, to analyze the two-step hybrid block method for zero-stability, the modulus of the roots of its first characteristic polynomial is expected to be simple or less than one as shown above. Thus, the corrector of the two-step hybrid block method are normalized to give the first characteristic polynomial as:

$$\left[ZA^0 - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right]$$ with

roots satisfying $\left| Z_j \right| \le 1$

The two-step hybrid block method is consistent if it has order $p \ge 1$ as satisfied above. Therefore, the two-step hybrid block method is convergent [6].

4. Numerical Results of the New Methods

In this section we presented some numerical results to compare our new hybrid block methods with other existing methods. The following notations are used in the tables of values:

1) 1SHBM: New One-Step Hybrid Block Method.
2) 2SHBM: New Two-Step Hybrid Block method.
3) Error = Computed solution Minus Exact Solution.
Problem 1: [10]
$y^{(1)} = 0.5(1 - y), y(0) = 0.5, h = 0.1$ with exact solution
$y(x) = 1 - 0.5e^{-0.5x}$
Problem 2: [10]

$y^{(1)} = -y, y(0) = 1, h = 0.1$ with exact solution $y(x) = e^{-x}$

Problem 3: [8]

In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In the preparation for winter weather, gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of $40 gal min^{-1}$ the well-mixed solution is pumped out at a rate of $40 gal min^{-1}$. Using a numerical integrator, how much of the additive is in the tank 0.1, 0.5 and 1 min after the pumping process begins?. Let y be the amount (in pounds) of additive in

the tank at time t. we know that y=100 when t=0. Thus, the initial value problem modeling the mixture process is;

$$y' = 80 - \frac{45}{(2000 - 5t)}, y(0) = 100, h = 0.1$$

With theoretical solution:

$$y(t) = 2(2000 - 5t) - \frac{3900}{(2000)^9}(2000 - 5t)^9$$

Table 1. Comparison of computed results for problem 1.

X	Exact solution	Computed solution (Zurni, et al., 2016)	Computed solution (1 SHBM)	Computed solution (2 SHBM)
0.1	0.52438528774964299546	0.52438528774960472804	0.52438528774964299544	0.52438528774964299544
0.2	0.54758129098202021342	0.54758129098194536511	0.54758129098202021339	0.54758129098202021339
0.3	0.56964601178747109638	0.56964601178736527269	0.56964601178747109634	0.56964601178747109634
0.4	0.59063462346100907066	0.59063462346087361956	0.59063462346100907061	0.59063462346100907061
0.5	0.61059960846429756588	0.61059960846413739010	0.61059960846429756581	0.61059960846429756581
0.6	0.62959088965914106696	0.62959088965895722513	0.62959088965914106688	0.62959088965914106688
0.7	0.64765595514064328282	0.64765595514044005788	0.64765595514064328272	0.64765595514064328272
0.8	0.66483997698218034963	0.66483997698195855368	0.66483997698218034951	0.66483997698218034951
0.9	0.68118592418911335343	0.68118592418887672320	0.68118592418911335330	0.68118592418911335330
1.0	0.69673467014368328820	0.69673467014343242661	0.69673467014368328806	0.69673467014368328806

Table 2. Comparison of errors for problem 1.

Computed solution in [11]	ERROR in [11]	ERROR (Zurni et al., 2016)	ERROR (1 SHBM)	ERROR (2 SHBM)
0.52438528774964238100	6.144610^{-16}	3.826740E-14	2.10^{-20}	2.10^{-20}
0.54758129098201904445	1.1689710^{-15}	7.484830E-14	3.10^{-20}	3.10^{-20}
0.56964601178746942845	1.6679310^{-15}	1.058240E-13	4.10^{-20}	4.10^{-20}
0.59063462346100695522	$2.11544\ 10^{-15}$	1.354510E-13	5.10^{-20}	5.10^{-20}
0.61059960846429505054	$2.51534\ 10^{-15}$	1.601760E-13	7.10^{-20}	7.10^{-20}
0.62959088965913819578	$2.87118\ 10^{-15}$	1.838420E-13	8.10^{-20}	8.10^{-20}
0.64765595514064009648	$3.18634\ 10^{-15}$	2.032250E-13	1.010^{-20}	1.010^{-20}
0.66483997698217688570	$3.46393\ 10^{-15}$	2.217960E-13	1.210^{-20}	1.210^{-20}
0.68118592418910964657	$3.70686\ 10^{-15}$	2.366300E-13	1.310^{-20}	1.310^{-20}
0.69673467014367937035	$3.91785\ 10^{-15}$	2.508620E-13	1.410^{-20}	1.410^{-20}

Table 3. Comparison of computed results for problem 2.

x	Exact solution	Computed solution (Zurni et al., 2016)	Computed solution (1 SHBM)	Computed solution (2 SHBM)
0.1	0.90483741803595957316	0.90483741804503260091	0.90483741803595957316	0.90483741803595957320
0.2	0.81873075307798185867	0.81873075309534995788	0.81873075307798185866	0.81873075307798185873
0.3	0.74081822068171786607	0.74081822070486153894	0.74081822068171786605	0.74081822068171786615
0.4	0.67032004603563930074	0.67032004606407889464	0.67032004603563930073	0.67032004603563930085
0.5	0.60653065971263342360	0.60653065974444846468	0.60653065971263342359	0.60653065971263342372
0.6	0.54881163609402643263	0.54881163612895298782	0.54881163609402643261	0.54881163609402643276
0.7	0.49658530379140951470	0.49658530382799175192	0.49658530379140951469	0.49658530379140951484
0.8	0.44932896411722215914	0.44932896415534885121	0.44932896411722159141	0.44932896411722159157
0.9	0.40656965974059911188	0.40656965977917485733	0.40656965974059911186	0.40656965974059911203
1.0	0.36787944117144232160	0.36787944121046227174	0.36787944117144232157	0.36787944117144232174

*Table 4. Comparison of errors for problem 2.*

Computed solution In [11]	ERROR In [11]	ERROR (Zurni et al., 2016)	ERROR (1 SHBM)	ERROR (2 SHBM)
0.90483741803610926985	1.496966910^{-13}	9.0730E-12	0.0	4.10^{-20}
0.81873075307825276100	2.709023310^{-13}	1.1768E-11	1.10^{-20}	6.10^{-20}
0.74081822068208554991	$3.6768384\ 10^{-13}$	2.3144E-11	2.10^{-20}	8.10^{-20}
0.67032004603608289288	$4.4359214\ 10^{-13}$	2.8440E-11	1.10^{-20}	1.110^{-20}
0.60653065971313514706	$5.0172346\ 10^{-13}$	3.1815E-11	1.10^{-20}	1.210^{-19}
0.54881163609457120641	$5.4477378\ 10^{-13}$	3.4927E-11	2.10^{-20}	1.310^{-19}
0.49658530379198460169	$5.7508699\ 10^{-13}$	3.6582E-11	1.10^{-20}	1.410^{-19}
0.44932896411781628883	$5.9469740\ 10^{-13}$	3.8127E-11	2.10^{-20}	1.410^{-19}
0.40656965974120447940	$6.0536752\ 10^{-13}$	3.8576E-11	2.10^{-20}	1.510^{-19}
0.36787944117205094291	$6.0862131\ 10^{-13}$	3.9020E-11	3.10^{-20}	1.410^{-20}

Table 5. Comparison of computed results and error for problem 3.

X	Exact solution	Computed solution ((2SEM) Zurni et al., 2016)	Computed solution (1 SHBM)	Computed solution (2 SHBM)
0.1	107.7662301168309486	107.76623267141251405	107.76623009862013152	107.76623014039012242
0.2	115.5149409193028512	115.51494346840455900	115.51494091641989830	115.51494096066439577
0.3	123.2461630508845221	123.24616814117862409	123.24616306468206404	123.24616314285139165
0.4	130.9599271090910725	130.95993218819786255	130.95992714090696968	130.95992729871549133
0.5	138.6562636455413535	138.65627125250773431	138.65626369669868680	138.65626393810552775
0.6	146.3352031660153396	146.33521075612409816	146.33520323782240227	146.33520356676826936
0.7	153.9967761305114566	153.99678623520317743	153.99677622426174555	153.99677664466866431
0.8	161.6410129533038516	161.64102303550463010	161.64101307027605861	161.64101358604710338
0.9	169.2679440029996051	169.26795658656269977	169.26794414445760852	169.26794475947662503
1.0	176.8775996025958863	176.87761215807155490	176.87759976978874276	176.87760048792006343

Table 6. Comparison of errors for problem 3.

x	ERROR (Zurni et al., 2016)	ERROR (1 SHBM)	ERROR (2 SHBM)
0.1	2.554000E-06	$3.061396905 \times 10^{-8}$	$2.355917392 \times 10^{-7}$
0.2	2.549000E-06	$4.990175158 \times 10^{-8}$	$5.815528062 \times 10^{-7}$
0.3	5.090000E-06	$7.053786702 \times 10^{-8}$	$1.0564381889 \times 10^{-7}$
0.4	5.079000E-06	$9.250752068 \times 10^{-8}$	$2.0458322997 \times 10^{-7}$
0.5	7.607000E-06	$1.1579590936 \times 10^{-7}$	$3.0877762970 \times 10^{-7}$
0.6	7.590000E-06	$1.4038822077 \times 10^{-7}$	$4.1819303026 \times 10^{-7}$
0.7	1.010000E-05	$1.6626963417 \times 10^{-7}$	$5.3279516340 \times 10^{-7}$
0.8	1.008000E-05	$1.9342531985 \times 10^{-7}$	$6.5254947373 \times 10^{-7}$
0.9	1.258000E-05	$2.21840443929 \times 10^{-7}$	$7.7742111078 \times 10^{-7}$
1.0	1.256000E-05	$2.5150014581 \times 10^{-7}$	$9.0737492156 \times 10^{-7}$
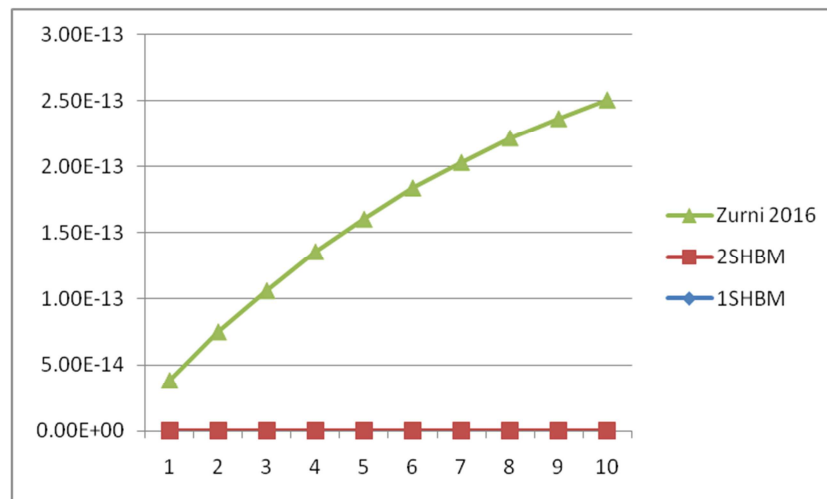
Figure 2. Error graph of problem 1.
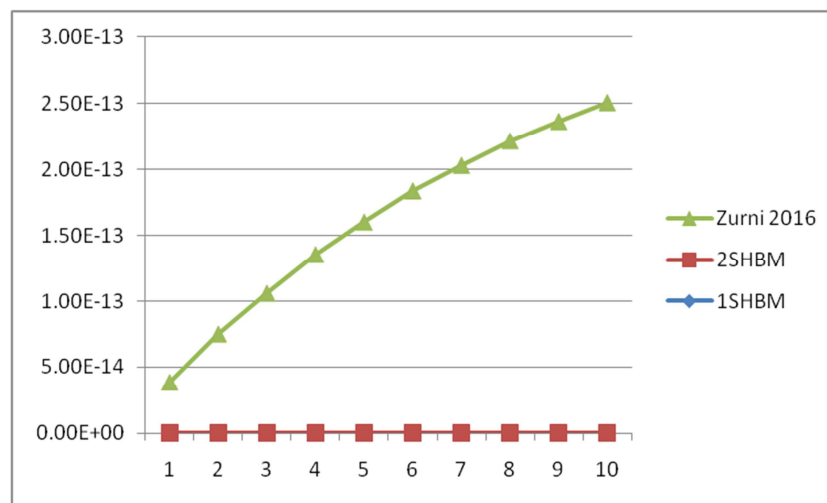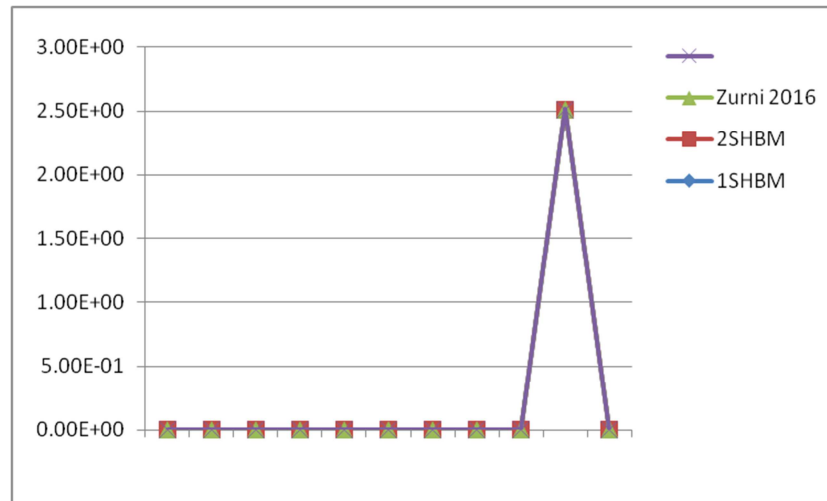

Figure 3. Error graph of problem 2.

Figure 4. *Error graph of problem 3.*

5. Conclusion

This article presented a new one and two-step hybrid block methods for solving first order ordinary differential equations. The new methods is seen to be quite flexible as the algorithm simultaneously produce block methods of step length k for solving first order ordinary differential equations. The new methods were seen to satisfy the basic properties to ensure convergence and their accuracy was also displayed which can be seen in the tables above. It was also observed, it is better to increase L [6] rather than the step k as shown in our numerical results above. Also, It was difficult to satisfy the zero-stability for larger k. Thus, this new methods are quite suitable for developing hybrid block methods for solving first order Ordinary Differential Equations.

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