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The Existence and Uniqueness Results of Solutions for a Fractional Hybrid Integro-differential System

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Abstract: This paper discuss two important results for a fractional hybrid boundary value problem of Riemann-Liouville integro-differential systems, the researches and the advance in this field and also the importance of this subject in the modeling of nonlinear real phenomena corresponding to a great variety of events gives the motivation to study this boundary value problem. The results are as follow, the first result consider the existence and uniqueness results of solutions for a fractional hybrid boundary value problem of Riemann-Liouville integro-differential system this result based on Krasnoslskii fixed point theorem for a sum of two operators, the second result is the uniqueness of solution for fractional hybrid boundary value problem of Riemann-Liouville integro-differential systems, the main result is based on Banach fixed point theorem, both results comes after transforming the system into Volterra integral system then transform again into operator system, then using fixed point theory to prove the results, this article was ended buy an example to well illustrat the results and ideas of proof.

Keywords: Hybrid Fixed Point Theorem, Banach Algebra, Operators Equations

1. Introduction

Nonlinear differential equations are crucial tools in the modeling of nonlinear real phenomena corresponding to a great variety of events, in relation with several fields of the physical sciences and technology. For instance, they appear in the study of the air motion or the fluids dynamics, electricity, electromagnetism, or the control of nonlinear processes, among others [1-8]. There solution of nonlinear differential equations requires, in general, the development of different techniques in order to deduce the existence and other essential properties of the solutions [9-15]. There are still many open problems related the solvability of nonlinear systems, apart form the fact that this is a field where advances are continuously taking place.

Perturbation techniques are useful in the nonlinear analysis for studying the dynamical systems represented by nonlinear differential and integral equations. Evidently, some differential equations representing a certain dynamical system have no analytical solution, so the perturbation of such problems can be helpful. The perturbed differential equations are categorized into various types. An important type of these such perturbations is called a hybrid differential equation (i.e. quadratic perturbation of a nonlinear differential equation). In [16] T. Bashiri et al. have considered the following non cooperative system with the fractional order $p \in (0, 1)$.

$$\begin{aligned} D^p\left[u(t) - f(t, u(t))\right] &= g\left(t, v(t), I^\alpha(v(t))\right) \\ D^p\left[v(t) - f(t, v(t))\right] &= g\left(t, u(t), I^\alpha(u(t))\right) \\ u(0) = 0, v(0) = 0 \quad &, \quad \alpha > 0, \end{aligned}$$

and investigated the existence of solutions. In [17] V. Daftardar-Gejji proposed the following fractional differential system given by

$$D^{\alpha_i} u_i = f_i\left(t, u_1, ..., u_n\right) \quad , \quad 0 < \alpha_i < 1, 1 < i < n,$$

and analysed the existence of positive solutions of the system in detail. In [18] S. Lui have considered the following cooperative system with the fractional order $\alpha, \beta \in (0,1)$

$$\begin{aligned} D^{\alpha}\left[u\left(t\right) - \Phi\left(t, u\left(t\right)\right)\right] &= f\left(t, u\left(t\right), v\left(t\right)\right) \\ D^{\beta}\left[v\left(t\right) - \Psi\left(t, v\left(t\right)\right)\right] &= g\left(t, u\left(t\right), v\left(t\right)\right) \\ u\left(0\right) = u\left(1\right) = 0 \quad &, \quad v\left(0\right) = v\left(1\right) = 0 \end{aligned}$$

In [25] B. Ahmed and all, study existence and uniqueness results for a nonlinear coupled system involving Caputo fractional derivatives with a new kind of coupled boundary conditions

$$\begin{aligned} D^{\alpha}u\left(t\right) &= f\left(t, u\left(t\right), v\left(t\right)\right) \\ D^{\beta}v\left(t\right) &= g\left(t, u\left(t\right), v\left(t\right)\right) \end{aligned}$$

$$(u+v)(0) = -(u+v)(T) \quad , \quad \int_{\eta}^{\epsilon} (u-v)(s)ds = A.$$

Where D^{χ} is the Caputo fractional derivative operator of order $\chi \in \alpha, \beta$, $\alpha, \beta \in (0,1]$. A is nonnegative constant, and $f, g : [0,T] \times \mathbb{R}^2 \mapsto \mathbb{R}$ are continuous functions.

Compared with the problem in [16], it is more partical to study the coupled system with different perturbation terms and nonlinearities. However, obviously, the mathematical model this case is more complex and more difficult to deal with in mathematics. Compared with the problem in [18] the perturbation terms is not that much complecated.

In this paper, we study the existence of solutions for a Dirichlet-type boundary value problem for the following fractional hybrid integro-differential system given by

$$D^{\alpha}\left[\frac{x\left(t\right)}{f_1\left(t, x\left(t\right)\right)}\right] = g_1\left(t, x\left(t\right), y\left(t\right)\right) + I^{\alpha-1}h_1\left(t, x\left(t\right), y\left(t\right)\right), \tag{1}$$

$$D^{\gamma}\left[\frac{y\left(t\right)}{f_2\left(t, y\left(t\right)\right)}\right] = g_2\left(t, x\left(t\right), y\left(t\right)\right) + I^{\gamma-1}h_2\left(t, x\left(t\right), y\left(t\right)\right), \tag{2}$$

$$x\left(0\right) = x\left(1\right) = 0, \quad y\left(0\right) = y\left(1\right) = 0, \tag{3}$$

where D^{α} denotes the Riemann-Liouville fractional derivative of order α, $(1 < \alpha \leqslant 2)$, D^{γ} denotes the Riemann-Liouville fractional derivative of order γ, $(1 < \gamma \leqslant 2)$, and I^{α} is the Riemann-Liouville fractional integral with order $\alpha > 0$, and I^{γ} is the Riemann-Liouville fractional integral with order $\gamma > 0$, $f_i \in C\left(J \times \mathbb{R}, \mathbb{R} \setminus \{0\}\right)$, $i = 1, 2$, and $g_i \in C\left(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\right)$, $i = 1, 2$ and $h_i \in C\left(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\right)$, $i = 1, 2$.

The rest of paper is organized as follows. In section 2, we recall some useful preliminaries. Section 3 contains the existence and uniqueness result which is obtained by means of Banach fixed point theorem. Section 4 contains the existence result which is obtained by means of Krasnoselskii fixed point theorem. Section 5 contains two exemples to illestrait our main results.

2. Preliminaries

For the convenience of the reader, we present here some necessary definitions from fractional calculus theory. These definitions and properties can be found in the recent monograph [20-25].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \mapsto \mathbb{R}$ is given by

$$I^{\alpha}f\left(t\right) = \frac{1}{\Gamma\left(\alpha\right)} \int_0^t \left(t-s\right)^{\alpha-1} f\left(s\right) ds, \tag{4}$$

provided that the right-hand side is defined pointwise, where $\Gamma\left(.\right)$ is the Gamma function.

Definition 2.2. Given a continuous function $f : (0, \infty) \longrightarrow \mathbb{R}$, its fractional derivative with order $\alpha > 0$ in the sense of Riemann-Liouville, is given

$$D^{\alpha}f\left(t\right) = \frac{1}{\Gamma\left(n-\alpha\right)} \left(\frac{d}{dt}\right)^n \int_0^t \left(t-s\right)^{n-\alpha-1} f\left(s\right) ds, \tag{5}$$

where $n = [\alpha] + 1$.

Lemma 2.1. [19] Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$. Then

$$I^{\alpha}D^{\alpha}u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2..., n$, where $n = [\alpha] + 1$

Lemma 2.2 (Ascoli-Arzela theorem). A be a subset of $C\left(J, E\right)$, A is relatively compact in $C\left(J, E\right)$ if and only if the following conditions are checked:

(i) The unit A is limited.

$\exists k > 0$ such that $\| f(x) \|_E \leq k$ for $x \in J$ and $f \in A$.

(ii) Unit A is equicontinuous.
$\forall \varepsilon > 0, \exists \delta > 0$ and for evry $t_1, t_2 \in J$ we have $|t_1, t_2| < \delta \Rightarrow \| f(t_1) - f(t_2) \|_E < \varepsilon$.

(iii) For any $x \in J$ the unit $\{f(x), f \in A\} \subset E$ is relatively compact.

Lemma 2.3 (Banach fixed point theorem). Let X be a non-empty complete metric space, and $T : X \mapsto X$ is a contraction mapping. Then, there exists a unique point $x \in X$ such that $Tx = x$.

Lemma 2.4 (Krasnoselskii fixed point theorem). Let E be a non-empty, bounded, closed and convex subset in Banach space X: If $A, B : E \mapsto E$satisfy the following assumptions:

1. $Ax + By \in E$, for every $x, y \in X$,
2. A is a contraction,
3. B is compact and continuous.

Then, there exists $z \in X$ such that $Az + Bz = z$.

3. Existence Result

Suppose that α, γ, and functions f_i, g_i, h_i, $i = 1, 2$ satisfy the problem (1) (2) (3). Then the unique solution of (1) (2) (3) is given by

Lemma 3.1.

$$\begin{aligned} x(t) &= f_1(t,x(t))\left[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x(s),y(s))ds + \int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x(s),y(s))ds\right] \\ &\quad - \left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x(s),y(s))ds + \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x(s),y(s))ds\right] t, \qquad (6) \\ y(t) &= f_2(t,y(t))\left[\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} g_2(s,x(s),y(s))ds + \int_0^t \frac{(t-s)^{2\gamma-2}}{\Gamma(2\gamma-1)} h_2(s,x(s),y(s))ds\right] \\ &\quad - \left[\int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} g_2(s,x(s),y(s))ds + \int_0^1 \frac{(1-s)^{2\gamma-2}}{\Gamma(2\gamma-1)} h_2(s,x(s),y(s))ds\right] t. \qquad (7) \end{aligned}$$

Proof. We apply the Riemann-Liouville fractional integral I^α and I^β on the both sides of (6) and (7) respectivelly, and using Lemma 2.1, we have

$$\begin{aligned} \frac{x(t)}{f_1(t,x(t))} &= I^\alpha g_1(t,x(t),y(t)) + I^{2\alpha-1} h_1(t,x(t),y(t)) + c_1 t + c_2, \\ \frac{y(t)}{f_2(t,y(t))} &= I^\gamma g_2(t,x(t),y(t)) + I^{2\gamma-1} h_2(t,x(t),y(t)) + c_3 t + c_4, \end{aligned}$$

then

$$x(t) = f_1(t,x(t))\left[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x(s),y(s))ds + \int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x(s),y(s))ds + c_1 t + c_2\right], \qquad (8)$$

$$y(t) = f_2(t,y(t))\left[\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} g_2(s,x(s),y(s))ds + \int_0^t \frac{(t-s)^{2\gamma-2}}{\Gamma(2\gamma-1)} h_2(s,x(s),y(s))ds + c_3 t + c_4\right], \qquad (9)$$

where $c_1, c_2 c_3$ and $c_4 \in \mathbb{R}$. Using the boundary value conditions, we fined that

$$\begin{aligned} c_1 &= -\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x(s),y(s))ds - \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x(s),y(s))ds, \\ c_2 &= 0, \\ c_3 &= -\int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)} g_2(s,x(s),y(s))ds - \int_0^1 \frac{(1-s)^{2\gamma-2}}{\Gamma(2\gamma-1)} h_2(s,x(s),y(s))ds, \\ c_4 &= 0. \end{aligned}$$

Substituting the values of c_1 and c_2 in (8), c_3 and c_4 in (9), we get solution (6) (7). The converse follows by direct computation. This completes the proof.

Our first result concerns the study of existence of solution for problem $(1-3)$ by using the Krasnoselskii fixed-point theorem. For this fact, we will need some assumptions about the functions f_i, g_i and h_i, $i = 1, 2$, previously defined.

Denote by $X = (C([0,1] \times \mathbb{R}) \times C([0,1] \times \mathbb{R}), \mathbb{R})$. The Banach space endowed with the norm $\|(x,y)\| = \|x\| + \|y| = \sup_{t\in[0,1]} |x(t)| + \sup_{t\in[0,1]} |y(t)|$, for $(x, y) \in X$.

H_1 The functions $f_i : J \times \mathbb{R} \longrightarrow \mathbb{R} \backslash \{0\}$, and $h_i : J \times \mathbb{R} \longrightarrow \mathbb{R}$, $i = 1, 2$ are continuous and there exist positives functions ϕ_i, ψ_i, $i = 1, 2$ with bounds $\|\phi_i\|$ and $\|\psi_i\|$ respectively, such that

$$|f_i(t,x) - f_i(t,y)| \leqslant \phi_i(t) |x - y|,$$

and

$$|g_i(t,x_1,y_1)-g_i(t,x_2,y_2)| \leqslant \chi_i(t)|x_1-x_2|+\kappa_i(t)|y_1-y_2|,$$
$$|h_i(t,x_1,y_1)-h_i(t,x_2,y_2)| \leqslant \psi_i(t)|x_1-x_2|+\varphi_i(t)|y_1-y_2|,$$

$i=1,2$ for all $t \in J$ and $x_i, y_i \in \mathbb{R}$.

H_2 There exists positive constants C_i, K_i, $i=1,2$, such that

$$|g_1(t,x_1,y_1)-g_1(t,x_2,y_2)| \leqslant C_1\|x_1-x_2\|+C_2\|y_1-y_2\|,$$

$$|g_2(t,x_1,y_1)-g_2(t,x_2,y_2)| \leqslant C_3\|x_1-x_2\|+C_4\|y_1-y_2\|,$$

$$|h_1(t,x_1,y_1)-h_1(t,x_2,y_2)| \leqslant K_1\|x_1-x_2\|+K_2\|y_1-y_2\|.$$

$$|h_2(t,x_1,y_1)-h_2(t,x_2,y_2)| \leqslant K_3\|x_1-x_2\|+K_4\|y_1-y_2\|.$$

H_3 There exists two constants $M_0, M_1>0$ such that

$$|h_1(t,x,y)| \leqslant M_0,$$

and

$$|h_2(t,x,y)| \leqslant M_1.$$

H_4 There exist two nonnegative functions $\mu_i, \eta_i \in L^1(J)$, $i=1,2$ such that for $(x,y) \in \mathbb{R}\times\mathbb{R}$ and $t \in J$.

$$|g_i(t,x(t),y(t))| \leqslant \mu_i(t), |h_i(t,x(t),y(t))| \leqslant \eta_i(t).$$

H_5 There exists a constant M such that

$$|f_i(t,u(t))| \leqslant M.$$

Theorem 3.1. Assume that the assumptions $(H_1)(H_4)$, and (H_5) hold. If

$$C\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(2\alpha)}\right)<1,$$
$$K\left(\frac{1}{\Gamma(\gamma+1)}+\frac{1}{\Gamma(2\gamma)}\right)<1$$

then the fractional integro-differential problem $(1)(2)(3)$ has at least one solution in $X=$ on J.

Proof. First, we will transform problem $(1)(2)(3)$ into a fixed point problem $Tx=x$, where T is the operator defined above. So, before starting the proof, we decompose T_i into a sum of two operators A_i and B_i, $i=1,2$ where

$$A_1(x,y)(t)=f_1(t,x(t))\left[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g_1(s,x(s),y(s))ds+\int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}h_1(s,x(s),y(s))ds\right],$$
$$A_2(x,y)(t)=f_2(t,y(t))\left[\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}g_2(s,x(s),y(s))ds+\int_0^t \frac{(t-s)^{2\gamma-2}}{\Gamma(2\gamma-1)}h_2(s,x(s),y(s))ds\right],$$

and

$$B_1(x,y)(t)=-\left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}g_1(s,x(s),y(s))ds+\int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}h_1(s,x(s),y(s))ds\right)t,$$
$$B_2(x,y)(t)=-\left(\int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)}g_2(s,x(s),y(s))ds+\int_0^1 \frac{(1-s)^{2\gamma-2}}{\Gamma(2\gamma-1)}h_2(s,x(s),y(s))ds\right)t.$$

Observe that

$$T_1(x,y)=A_1(x,y)+B_1(x,y),$$
$$T_2(x,y)=A_2(x,y)+B_2(x,y).$$

Now, we show that the operators A_1, A_2 and B_1, B_2 satisfy all the conditions of Lemma 2,4 in a series of steps.
Step 1. We define the set

$$\Omega = \{(x,y) \in X, \|(x,y)\|_X \leqslant \rho\}$$

where ρ is a positive real constant chosen so that

$$(M+1)\left(\frac{\|\mu_1\|}{\Gamma(\alpha+1)} + \frac{\|\eta_1\|}{\Gamma(2\alpha)}\right) \leqslant \rho \tag{10}$$

and we show that $A_i + B_i \in \Omega$. So, for $(x,y) \in \Omega$ and $t \in J$, we have

$$\begin{aligned}
& |A_1(x,y)(t) + B_1(x,y)(t)| \\
\leqslant\ & |f_1(t,x(t))\left[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x(s),y(s))ds + \int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x(s),y(s))ds\right] \\
& - \left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x(s),y(s))ds + \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x(s),y(s))ds\right] t| \\
\leqslant\ & |f_1(t,x(t))| \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |g_1(s,x(s),y(s))|\, ds + \int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} |h_1(s,x(s),y(s))|\, ds\right) \\
& + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |g_1(s,x(s),y(s))|\, ds + \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} |h_1(s,x(s),y(s))|\, ds \\
\leqslant\ & M\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \mu_1(t)ds + \int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} \eta_1(t)ds\right) \\
& + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \mu_1(t)ds + \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} \eta_1(t)ds \\
\leqslant\ & M\left(\frac{\|\mu_1\|}{\Gamma(\alpha+1)} + \frac{\|\eta_1\|}{\Gamma(2\alpha)}\right) + \left(\frac{\|\mu_1\|}{\Gamma(\alpha+1)} + \frac{\|\eta_1\|}{\Gamma(2\alpha)}\right) \\
\leqslant\ & (M+1)\left(\frac{\|\mu_1\|}{\Gamma(\alpha+1)} + \frac{\|\eta_1\|}{\Gamma(2\alpha)}\right) \\
\leqslant\ & \rho.
\end{aligned}$$

Thus, $\|A_1(x,y) + B_1(x,y)\|_X \leqslant \rho$ which means that $A_1(x,y) + B_2(x,y) \in \Omega$.
Analogously, we can obtain

$$\begin{aligned}
& |A_2(x,y)(t) + B_2(x,y)(t)| \\
\leqslant\ & (M+1)\left(\frac{\|\mu_2\|}{\Gamma(\gamma+1)} + \frac{\|\eta_2\|}{\Gamma(2\gamma)}\right) \\
\leqslant\ & \rho.
\end{aligned}$$

Thus, $\|A_2(x,y) + B_2(x,y)\|_X \leqslant \rho$ which means that $A_2(x,y) + B_2(x,y) \in \Omega$.
Step 2. B_i is a contraction on Ω. From the definition of the operators B_i, $i = 1, 2$, we have for $(x_1, y_1), (x_2, y_2) \in \Omega$, and $t \in J$

$$\begin{aligned}
& |B_1(x_1,y_1)(t) - B_1(x_2,y_2)(t)| \\
\leqslant\ & |\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x_1(s),y_1(s))ds + \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x_1(s),y_1(s))ds \\
& - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x_2(s),y_2(s))ds + \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x_2(s),y_2(s))ds| \\
\leqslant\ & \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |g_1(s,x_1(s),y_1(s)) - g_1(s,x_2(s),y_2(s))|\, ds \\
& + \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} |h_1(s,x_1(s),y_1(s)) - h_1(s,x_2(s),y_2(s))|\, ds \\
\leqslant\ & \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} C_1(x_1(s) - x_2(s)) + C_2(y_1(s) - y_2(s))ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} C_1(x_1(s)-x_2(s))+C_2(y_1(s)-y_2(s))ds \\
\leqslant\ & \frac{C_1\|x_1-x_2\|+C_2\|y_1-y_2\|}{\Gamma(\alpha+1)}+\frac{C_1\|x_1-x_2\|+C_2\|y_1-y_2\|}{\Gamma(2\alpha)} \\
\leqslant\ & C\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(2\alpha)}\right)(\|x_1-x_2\|+\|y_1-y_2\|) \\
\leqslant\ & C\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(2\alpha)}\right)\|(x_1-x_2,y_1-y_2)\|
\end{aligned}
$$

Analogously, we can obtain

$$
\begin{aligned}
& |B_2(x_1,y_1)(t)-B_2(x_2,y_2)(t)| \\
\leqslant\ & K\left(\frac{1}{\Gamma(\gamma+1)}+\frac{1}{\Gamma(2\gamma)}\right)\|(x_1-x_2,y_1-y_2)\|
\end{aligned}
$$

Hence, from (4.1), it follows that B_i, $i=1,2$ is a contraction on Ω.

Step 3. A_i is completely continuous on Ω. Then we show that $(A_i\Omega)$ is uniformly bounded, $\overline{(A_i\Omega)}$ is equi-continuous, and $A_i:\Omega\mapsto\Omega$ is continuous.

For $(x,y)\in\Omega$ and $t\in J$, we have

$$
\begin{aligned}
|A_1(x,y)(t)| &\leqslant \left|f_1(t,x(t))\left[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x(s),y(s))ds+\int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x(s),y(s))ds\right]\right| \\
&\leqslant |f_1(t,x(t))|\left[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|g_1(s,x(s),y(s))|ds+\int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}|h_1(s,x(s),y(s))|ds\right] \\
&\leqslant M\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\mu_1(s)ds+\int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\eta_1(s)ds\right) \\
&\leqslant M\left(\frac{\|\mu_1\|}{\Gamma(\alpha+1)}+\frac{\|\eta_1\|}{\Gamma(2\alpha)}\right)
\end{aligned}
$$

Analogously, we can obtain

$$
|A_2(x,y)(t)|\leqslant M\left(\frac{\|\mu_2\|}{\Gamma(\gamma+1)}+\frac{\|\eta_2\|}{\Gamma(2\gamma)}\right)
$$

Then, $(A_i\Omega)$ is uniformly bounded. Now we show that $\overline{(A_i\Omega)}$ is equicontinuous

Let $t_1,t_2\in J$ with $t_1<t_2$ we have for any $(x,y)\in\Omega$

$$
\begin{aligned}
& |A_1(x,y)(t_2)-A_1(x,y)(t_1)| \\
\leqslant\ & |f_1(t_2,x(t_2))\left[\int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x(s),y(s))ds+\int_0^{t_2} \frac{(t_2-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x(s),y(s))ds\right] \\
& -f_1(t_1,x(t_1))\left[\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x(s),y(s))ds+\int_0^{t_1} \frac{(t_1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x(s),y(s))ds\right]| \\
\leqslant\ & M\Bigg[\int_0^{t_2}\left(\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\right)|g_1(s,x(s),y(s))|ds \\
& +\int_0^{t_2}\left(\frac{(t_2-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}-\frac{(t_1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\right)|h_1(s,x(s),y(s))|ds \\
& +\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}|g_1(s,x(s),y(s))|ds+\int_{t_1}^{t_2} \frac{(t_1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}|h_1(s,x(s),y(s))|ds\Bigg] \\
\leqslant\ & M\Bigg[\|\mu_1\|\int_0^{t_2}\left(\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\right)ds+\|\eta_1\|\int_0^{t_2}\left(\frac{(t_2-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}-\frac{(t_1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\right)ds \\
& +\|\mu_1\|\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\|\eta_1\|\int_{t_1}^{t_2} \frac{(t_1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}ds\Bigg].
\end{aligned}
$$

Analogously, we can obtain

$$\begin{aligned}
&|A_1(x,y)(t_2)-A_1(x,y)(t_1)|\\
\leqslant\ & M\bigg[\|\mu_2\|\int_0^{t_2}\left(\frac{(t_2-s)^{\gamma-1}}{\Gamma(\gamma)}-\frac{(t_1-s)^{\gamma-1}}{\Gamma(\gamma)}\right)ds+\|\eta_2\|\int_0^{t_2}\left(\frac{(t_2-s)^{2\gamma-2}}{\Gamma(2\gamma-1)}-\frac{(t_1-s)^{2\gamma-2}}{\Gamma(2\gamma-1)}\right)ds\\
&+\|\mu_2\|\int_{t_1}^{t_2}\frac{(t_2-s)^{\gamma-1}}{\Gamma(\gamma)}ds+\|\eta_2\|\int_{t_1}^{t_2}\frac{(t_1-s)^{2\gamma-2}}{\Gamma(2\gamma-1)}ds\bigg].
\end{aligned}$$

which is independent of $(x,y)\in\Omega$. As $t_1\longrightarrow t_2$, the right-hand side of the obove inequalities tend to zero. Therefore, it follows that $\overline{(A_i\Omega)}$ is equicontinuous.

Step 4. Finally we show that the operators A_1, A_2 are continuous in X. Let $\{(x_n,y_n)\}$ be a sequance in Ω converging to a point $(x,y)\in\Omega$. Then by Lebesgue domination convergence theorem, for all $t\in J$ we obtain

$$\begin{aligned}
&\lim_{n\mapsto\infty}A_1(x_n,y_n)(t)\\
=\ &\lim_{n\mapsto\infty}f_1(t,x_n(t))\left[\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g_1(s,x_n(s),y_n(s))ds+\int_0^t\frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}h_1(s,x_n(s),y_n(s))ds\right]\\
=\ &\lim_{n\mapsto\infty}f_1(t,x_n(t))\left[\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\lim_{n\mapsto\infty}g_1(s,x_n(s),y_n(s))ds+\int_0^t\frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}\lim_{n\mapsto\infty}h_1(s,x_n(s),y_n(s))ds\right]\\
=\ &f_1(t,x(t))\left[\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g_1(s,x(s),y(s))ds+\int_0^t\frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}h_1(s,x(s),y(s))ds\right]\\
=\ &A_1(x,y)(t).
\end{aligned}$$

Same prove for the operator A_2.

Consequently, A_i is continuous. Therefore, A_i is also relatively compact on Ω.

Owing to the Arzela-Ascoli theorem, it follows that A_i is compact on Ω. Then by Krasnoselskii's fixed-point theorem, the operator A_i+B_i has a fixed point in Ω. Finally, we deduce that problem (1-3) has at least one solution in X on J.

4. Existence and Uniqueness Result

This section is devoted to the study of existence and uniqueness of solution of problem $(1-3)$ using Banach fixed-point theorem.

Theorem 4.1. Assume that the assumptions (H1), (H2), and (H3) hold, then the fractional integro-differential system $(1-3)$ has a unique solution in X on J.

Proof In view of Lemma 3.1 we introduce an operator $T: X\mapsto X$ associated with the problem $(1-3)$ as follows

$$T(x,y)(t)=(T_1(x,y)(t),T_2(x,y)(t)),\tag{11}$$

$$\begin{aligned}
T_1(x,y)(t)\ =\ &f_1(t,x(t))\left[\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g_1(s,x(s),y(s))ds+\int_0^t\frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}h_1(s,x(s),y(s))ds\right]\\
&-\left[\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}g_1(s,x(s),y(s))ds+\int_0^1\frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)}h_1(s,x(s),y(s))ds\right]t, \qquad (12)\\
T_2(x,y)(t)\ =\ &f_2(t,y(t))\left[\int_0^t\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}g_2(s,x(s),y(s))ds+\int_0^t\frac{(t-s)^{2\gamma-2}}{\Gamma(2\gamma-1)}h_2(s,x(s),y(s))ds\right]\\
&-\left[\int_0^1\frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)}g_2(s,x(s),y(s))ds+\int_0^1\frac{(1-s)^{2\gamma-2}}{\Gamma(2\gamma-1)}h_2(s,x(s),y(s))ds\right]t. \qquad (13)
\end{aligned}$$

Now, we show that the operator T has a fixed point in B_r which represents the unique solution of our problem $(1-3)$. So, the proof is down in two steps.

Step 1. We will show that $T_iB_r\subset B_r$, $i=1,2$. We get for each $t\in J$ and $x\in B_r$

$$\begin{aligned}
&|T_1(x,y)(t)| \\
\leqslant\ & |f_1(t,x(t))\left[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x(s),y(s))ds + \int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x(s),y(s))ds\right] \\
&- \left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x(s),y(s))ds + \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x(s),y(s))ds\right] t| \\
\leqslant\ & |f_1(t,x(t))| \\
&\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x(s),y(s))ds + \int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x(s),y(s))ds\right| \\
&+ \left|\left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x(s),y(s))ds + \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x(s),y(s))ds\right) t\right| \\
\leqslant\ & \left(|f_1(t,x(t)) - f_1(t,0)| + |f_1(t,0)|\right) \\
&\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |g_1(s,x(s),y(s))|\, ds + \int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} |h_1(s,x(s),y(s))|\, ds \\
&+ \left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |g_1(s,x(s),y(s))|\, ds + \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} |h_1(s,x(s),y(s))|\, ds\right) t \\
\leqslant\ & (r_1 \|\phi_1\| + F_{10}) \\
&\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\varphi_1(x,y)ds + \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} \left|(h_1(s,x(s),y(s)) - h_1(s,0,0)| + |h_1(s,0,0)|\right) ds \\
&+ \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\varphi_1(x,y)ds + \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} \left|(h_1(s,x(s),y(s)) - h_1(s,0,0)| + |h_1(s,0,0)|\right) ds \\
\leqslant\ & (r_1 \|\phi_1\| + F_{10} + 1) \\
&\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\varphi_1(x,y)ds + \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} \left|(h_1(s,x(s),y(s)) - h_1(s,0,0)| + |h_1(s,0,0)|\right) ds \\
\leqslant\ & (r_1 \|\phi_1\| + F_{10} + 1)\left(\frac{\|u\|\,\varphi_1(x,y)}{\Gamma(\alpha+1)} + \frac{\|v\|\,\psi_1(x,y) + M_0}{\Gamma(2\alpha)}\right).
\end{aligned}$$

which leads to

$$\begin{aligned}
\|T_1(x,y)\| &\leqslant (r_1 \|\phi_1\| + F_1 + 1)\left(\frac{\|u\|\,\varphi_1(x,y)}{\Gamma(\alpha+1)} + \frac{\|v\|\,\psi_1(x,y) + M_0}{\Gamma(2\alpha)}\right) \\
&< r.
\end{aligned}$$

Therefore $T_1(x,y) \leqslant r$, which means that $T_1 B_r \subset B_r$
In the same way, for $(x,y) \in B_r$, one can obtain

$$\begin{aligned}
\|T_2(x,y)\| &\leqslant (r_2 \|\phi_2\| + F_2 + 1)\left(\frac{\|u\|\,\varphi_2(x,y)}{\Gamma(\gamma+1)} + \frac{\|v\|\,\psi_2(x,y) + M_0}{\Gamma(2\gamma)}\right) \\
&< r.
\end{aligned}$$

Therefore, for any $(x,y) \in B_r$, we have

$$\begin{aligned}
\|T(x,y)| &= \|T_1(x,y)\| + \|T_2(x,y)\| \\
&\leqslant (r_1 \|\phi_1\| + F_1 + 1)\left(\frac{\|u\|\,\varphi_1(x,y)}{\Gamma(\alpha+1)} + \frac{\|v\|\,\psi_1(x,y) + M_0}{\Gamma(2\alpha)}\right) \\
&\quad + (r_2 \|\phi_2\| + F_2 + 1)\left(\frac{\|u\|\,\varphi_2(x,y)}{\Gamma(\gamma+1)} + \frac{\|v\|\,\psi_2(x,y) + M_0}{\Gamma(2\gamma)}\right) < r.
\end{aligned}$$

which shows that T maps B_r into itself.

Step 2. We will show that $T : B_r \mapsto B_r$ is a contraction.

In order to show that the operator T is a contraction, let $(x_1, y_1), (x_2, y_2) \in X$ and $t \in [0,1]$. Then, in view of (H2), we obtain

$$\begin{aligned}
& |T_1(x_1,y_1) - T_1(x_2,y_2)| \\
\leqslant\; & |f_1(t,x_1(t))\left[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x_1(s),y_1(s))ds + \int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x_1(s),y_1(s))ds\right] \\
& -\left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x_1(s),y_1(s))ds + \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x_1(s),y_1(s))ds\right] t \\
& -f_1(t,x_2(t))\left[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x_2(s),y_2(s))ds + \int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x_2(s),y_2(s))ds\right] \\
& +\left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,x_2(s),y_2(s))ds + \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} h_1(s,x_2(s),y_2(s))ds\right] t| \\
\leqslant\; & (r_1\|\phi_1\| + F_{10}) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |g_1(s,x_1(s),y_1(s)) - g_1(s,x_2(s),y_2(s))|\, ds \\
& + \int_0^t \frac{(t-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} |h_1(s,x_1(s),y_1(s)) - h_1(s,x_2(s),y_2(s))|\, ds \\
& + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |g_1(s,x_1(s),y_1(s)) - g_1(s,x_2(s),y_2(s))|\, ds \\
& + \int_0^1 \frac{(1-s)^{2\alpha-2}}{\Gamma(2\alpha-1)} |h_1(s,x_1(s),y_1(s)) - h_1(s,x_2(s),y_2(s))|\, ds
\end{aligned}$$

we conclude that T is a contraction. Thenby Lemma 2.1, a unique point $X \in X$ exists such that $Tx = x$. It is the unique solution of our BVP $(1-3)$.

5. Example

Consider the following BVP of fractional integro-differential system:

$$\begin{aligned}
D^{\frac{3}{2}}\left[\frac{x(t)}{t^2\cos(|x(t)|)}\right] &= \frac{1+t^2+\sin(x(t))+\cos(y(t))}{8(1+t)} \\
& +\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-s)^{-\frac{1}{2}} \frac{s^2+s+1+\sin(x(s))+\cos(y(s))}{8(1+s)(1+s^2)} ds,
\end{aligned} \tag{14}$$

$$\begin{aligned}
D^{\frac{4}{3}}\left[\frac{y(t)}{t^2\sin(|y(t)|)}\right] &= \frac{(1+t)(1+t^2)+\cos(x(t))+\sin(y(t))}{(1+t^3)(1+t^2)} \\
& +\frac{1}{\Gamma\left(\frac{1}{3}\right)}\int_0^t (t-s)^{-\frac{2}{3}} \frac{s+1+|x(s)|+|y(s)|}{8(1+s)(1+s^2)} ds,
\end{aligned} \tag{15}$$

$$x(0) = x(1) = 0 \quad , \quad y(0) = y(1) = 0, \tag{16}$$

The problem (14-16) is a particular case of (1-3) with $\alpha = \frac{3}{2}, \gamma = \frac{4}{3}$ and

$$f_1(t,x(t)) = t^2\cos(|x(t)|) \quad , \quad f_2(t,y(t)) = t^2\sin(|y(t)|),$$

$$g_1(t,x(t),y(t)) = \frac{1+t^2+\sin(x(t))+\cos(y(t))}{8(1+t)} \quad , \quad g_2(t,x(t),y(t)) = \frac{(1+t)(1+t^2)+\cos(x(t))+\sin(y(t))}{(1+t^3)(1+t^2)},$$

and

$$h_1(t,x(t),y(t)) = \frac{t^2+t+1+\sin(x(t))+\cos(y(t))}{8(1+t)(1+t^2)} \quad , \quad h_2(t,x(t),y(t)) = \frac{t+1+|x(t)|+|y(t)|}{8(1+t)(1+t^2)}.$$

Clearly f_i, g_i and h_i, $i = 1, 2$ are continuous functions and satisfy the condition (H_1) with $\phi_1 = t^2$, $\phi_2 = t^2$ and

$$|g_1(t, x_1, y_1) - g_1(t, x_2, y_2)| \leqslant \frac{1}{8(1+t)}|x_1 - x_2| + \frac{1}{8(1+t)}|y_1 - y_2|,$$
$$|g_2(t, x_1, y_1) - g_2(t, x_2, y_2)| \leqslant \frac{1}{(1+t^3)(t+t^2)}|x_1 - x_2| + \frac{1}{(1+t^3)(t+t^2)}|y_1 - y_2|,$$

then

$$\chi_1(t) = \frac{1}{8(1+t)} \quad , \quad \chi_2(t) = \frac{1}{(1+t^3)(t+t^2)},$$
$$\kappa_1(t) = \frac{1}{8(1+t)} \quad , \quad \kappa_2(t) = \frac{1}{(1+t^3)(t+t^2)},$$

and

$$|h_1(t, x_1, y_1) - h_1(t, x_2, y_2)| \leqslant \frac{1}{8(1+t)(1+t^2)}|x_1 - x_2| + \frac{1}{8(1+t)(1+t^2)}|y_1 - y_2|,$$
$$|h_2(t, x_1, y_1) - h_2(t, x_2, y_2)| \leqslant \frac{1}{8(1+t)(1+t^2)}|x_1 - x_2| + \frac{1}{8(1+t)(1+t^2)}|y_1 - y_2|,$$

then

$$\psi_1(t) = \frac{1}{8(1+t)(1+t^2)} \quad , \quad \psi_2(t) = \frac{1}{8(1+t)(1+t^2)}$$
$$\varphi_1(t) = \frac{1}{8(1+t)(1+t^2)} \quad , \quad \varphi_2(t) = \frac{1}{8(1+t)(1+t^2)}$$

Since the assumptions $(H_1) - (H_5)$ hold, according to Theorem 3.1 the BVP has at least one solution.

To see if the solution is unique, note that assumptions $(H_1) - (H_5)$ are hold, from first part of existence results. Also, the condition of Theorem 3.2 satisfied, therefore from Theorem 3.2 the BVP has a unique solution.

6. Conclusions

In this work, we consider the existence results for a nonlocal bounbary value problem of Caputo-type Hadamard hybrid fractional integro-differential equations. The problem contain two different types of perturbalion, this work based on fixed point theorey and fractional calculus and the work was done as follow, by transforming the problem into a Volttera integral system and using the Krasnoselskii fixed point theorem, we get the existence results of solutions for the boundary value problem (1) under some conditions. Then, using the Banach fixed point theorem, we get the existence and uniqueness of solution for the boundary value problem, after transforming the problem into a fixed point problem.

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Conflicts of Interest

The authors declare no conflicts of interest.

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