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Communication

Assessment of Zitterbewegung Interpretation for Free Particle Solution Using the Concept of Relativistic Wave

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Abstract

The Assessment of the interpretation of Zitterbewegung for free particle solution using various models has been carried out in this work where we firstly considered by using the free particle solution for which Dirac equation and projection operator were carried out.. ln this case, it was invariably revealed that the solution have two doubly degenerate eigenvalues representing positive and negative state that was further support by the analysis which was carried out using Heisenberg’s equation and the representation in relativistic velocity and semi-classical equation of motion for acceleration., but in this second case, the results obtained were observed to have two terms, first terms standing in for rapidly oscillatory motion and the second term representing average motion of the particle in x-direction. The first term in the expression is the one deemed to signify zitterbewegung which is one considered to be sas a result of the fluctuation resulting from the interference between negative and positive energy state while the other term is the normal state terms signifying the average motion of the particle. This therefore confirms the explicit nature of using Dirac equation in handling problems involving the behavior of free particles in field as compared to the use of semiclassical equation of motion.

Keywords

Analysis, Zitterbewegung, Free Particle, Dirac Equation, Heisenberg’s Representation, Relativistic Velocity, Fluctuation, Oscillatory Motion, Eigenvalues

1 Introduction

The issue of rapidly oscillatory motion of particles within the domain of relativistic wave equation has been of great concern to so many physicists when it came to solution Dirac equation for free particle [1-5] and based on the analysis of wave packet solution from the Dirac for relativistic electron as a particle in free space, it has inferred that there is a fluctuation around the medium due to the interference produced between positive and negative energies that are obtained when Dirac equation is solved which brings about jittery motion that is referred as Zitterbewegung (ZBW) [6-8]. ZBW has been interpreted to be as a result of interference between positive and negative energy wave component that results from the solution. Some has commented that it is as a result of interaction of electrons with spontaneous formation and annihilation of electron-positron pair [9, 10] while it is infer that it is due to electron spin and magnetic moment generated by locally circulatory motion of electrons which lots of physicists have proposed independently and presented in different format that could make incursion in interpreting the proper picture of ZBW. Schrodinger gave his own ZBW model for mo-

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tion of electron ground state in assessing and analyzing the solution of Dirac equation [11-13]. It was apparently over looked as some assumed that it was not a real physical concept because it could not have any regard to it, but some Physicists revisited the idea of zbw and used theoretical approach in specific case such as electron [14, 15] modification of position operator was introduced in some other approach to project the system to dynamical invariant subspace of quantum mechanical Hilbert space associated with the anti-particles [16] in order to highlight clear picture apart from the explanation given by Schrodinger as to the existence of ZBW it did not yield any result not until it was paradoxically stemmed that the ZBW is actually due to co-existence of particle and antiparticle which likened to pair creation and annihilation) Further study by Niehaus on ZBW and electron and others who have also used various models and different perspective have enhanced the concept of ZBW [17] with statistical model explained the reality of ZBW with support from substructure of electron as illustrated in the solution of hydrogen atom in which negative energy appeared showing the existence of ZBW in the solution. This therefore relates the interaction of electron with the potential of the nucleus of the atom which the gives raise to some finite amplitude for the formation of an electron –positron pair.

In this work, we intend to analyze the interpretation of zitterbewegung occurrence in solution of Dirac Equation for free particle by exploring various aspect of solution that involve Dirac Hamiltonian

2. Theoretical Procedures

Here we present the theoretical presentation by first of all looking at the form of Dirac Equation and the results of the solutions

A. Dirac Equation

Dirac derived his relativistic wave equation in relation to the Schrodinger form

$$i\hbar\frac{\partial\psi}{\partial t}=\widehat{H}\psi \tag{1}$$

with positive definite probability density while Dirac on his own constructed his using Hamiltonian that is linear in spatial derivatives (18) resulting to

$$i\hbar\frac{\partial\psi}{\partial t}\left[\frac{hc}{i}\left(\widehat{\alpha}_1\frac{\partial}{\partial x^1}+\widehat{\alpha}_2\frac{\partial}{\partial x^2}+\widehat{\alpha}_3\frac{\partial}{\partial x^3}\right)+\widehat{\beta}m_oc^2\right]\psi=\widehat{H}_f\psi \tag{2}$$

With its counterpart for Maxwell equation given as

$$-\frac{1}{i}\left(\widehat{\alpha}\,{}^1\!/_c\frac{\partial}{\partial t}\widehat{\alpha}^1\frac{\partial}{\partial x}+\widehat{\alpha}^2\frac{\partial}{\partial y}+\widehat{\alpha}^3\frac{\partial}{\partial z}\right)\psi=\frac{4\pi}{c}\varphi \tag{3}$$

With the relations

$$\alpha_i\alpha_k=2\delta_{ki}\ \text{ and }\ \alpha_k\beta+\beta\alpha_i=0\ \ (\text{k, i} =1,\ 2,\ 3) \tag{4}$$

Define the Dirac algebra that is realized by 4x 4 matrices

$$\alpha=\begin{pmatrix}0&\sigma\\ \sigma&0\end{pmatrix}\ \text{ and }\ \beta=\begin{pmatrix}\mathrm{I}_2&0\\ 0&\mathrm{I}_2\end{pmatrix} \tag{5}$$

Where $\alpha=(\sigma_1\sigma_2\sigma_3\sigma_4)$ the vector of Pauli is spin matrices and I_n denotes the nxn unit matrix

Each of these equations have very interesting characteristic of time domain that has been commonly notable in quantum mechanics because it has been common for such an equation as Schrodinger or Dirac equation that describe system or environment to have a corresponding time dependent equation in a disentangling approximated manner such that the motion of the system or the environment provides a time derivative to monitor the development of the system [19-21] and invariably when solved gives a picture of the behavior of particle in them.

B. Free particle Solution of Dirac Equation

Therefore proceeding to the free particle solution of Dirac equation, we first of all state the relativistic free particle Dirac Hamiltonian that gives the base point to kick start the solution as

$$\widehat{H}=i\hbar c\alpha.p+\beta m_oc^2 \tag{6}$$

Considering the stationary state with ansatz, equation 4 is transformed into

$$\varepsilon\psi(x)=\widehat{H}_f\psi(x) \tag{7}$$

Where the term ε and $\psi(x)$ describe the time evolution and stationary state respectively

By using the explicit form of equation (5), equation (6) which is realizable by splitting the four components of the evolved wave function, ψ into

$$\varphi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \text{ and } \chi = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} \text{ we have equation of the form } \varepsilon\psi = c\hat{\sigma}.\hat{p}\chi + m_o c^2 \psi and \varepsilon\psi = c\hat{\sigma}.\hat{p}\chi - m_o c^2 \quad (8)$$

Again if we considered the state with definite momentum, we have

$$\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} \varphi_o \\ \chi_o \end{pmatrix} \exp\left[i/\hbar \right] p.x \quad (9)$$

the counterparts of equation (8) could be of the form

$$\left(\varepsilon - m_o c^2\right)\mathrm{II}\varphi_o - c\hat{\sigma}.p\chi_o = 0 \text{ and}$$

$$-c\hat{\sigma}.p\varphi_o + \left(\varepsilon + m_o c^2\right)\mathrm{II}\chi_o \quad (10)$$

However, this being linearly homogeneous system of equations for φ_o and χ_o respectively, they have nontrivial solutions except in a case of a vanishing determinant of the coefficients, that is

$$\begin{vmatrix} (\varepsilon - m_0 c^2)\mathrm{II} & -c\hat{\sigma}.p \\ -c\hat{\sigma}.p & (\varepsilon - m_0 c^2)\mathrm{II} \end{vmatrix} = 0 \quad (11)$$

$$\left(\varepsilon - m_o^2 c^4\right)\mathrm{II} - c^2\left(\hat{\sigma}.p\right)\left(\hat{\sigma}.p\right) = 0$$

$\left(\varepsilon - m_o^2 c^4\right)\mathrm{II} - c^2\left(\hat{\sigma}.p\right)\left(\hat{\sigma}.p\right) = 0$ Using this relation; $\left(\sigma.A\right)\left(\sigma.B\right) = A.B\mathrm{II} + i\hat{\sigma}.\left(AxB\right)$, equation (10) is transformed into

$$\left(\varepsilon - m_o^2 c^4\right)\mathrm{II} - c^2\left(\hat{\sigma}.p\right)\left(\hat{\sigma}.p\right) = 0,$$

$$\varepsilon^2 = m_o^2 c^4 + c^2 p^2 \quad (12)$$

From equation (11), we obtain expression that is as follows [18]

$$\varepsilon = \pm E_p, \quad E_p = \pm c\sqrt{(p^2 + m_o^2 c^2} \quad (13)$$

From equation 12, there are two signs time-evolution factors for ε which signify two types of solutions amenable to the Dirac equation which obviously represent the idea of positive and negative solutions respectively. (5)

For a fixed ε, we obtain from equation (11)

$$\chi_o = \frac{c\left(\hat{\sigma}.p\right)}{m_o^2 c^2 + \varepsilon}\varphi_o \quad (14)$$

And by denoting φ_o in the form

$$\varphi_o = u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (15)$$

And with normalization

$$u^\dagger u = u_1^* u + u_2^* u = 1 \quad (16)$$

Where u_1, u_2 are complex.

Using equations (7) and (9) a complete set of positive and negative free solution Dirac equation emerges as

$$\psi_{p\lambda}(x,t) = N\left(\frac{\overset{U}{c(\hat{\sigma}.\mathrm{p})}}{m_o c^2 + \lambda p}u\right)\frac{\exp\left[i(x.p - \overbrace{\lambda E_p t}^{c})/\hbar\right]}{\sqrt{2\pi\hbar}^3} \quad (17)$$

In accordance with time evolution factor $\varepsilon = \lambda E_p$ where $\lambda = \pm 1$ yielding $\varepsilon = +E_p$ and $\varepsilon = -E_p$ for $\lambda = +$ and $\lambda = -$ respectively which depict that there is presence of negative value of energy.

In a similar manner, if the energy to the free particle solution is assessed in he Canonical Formalism using the integral $E = \int_v T^o{}_o d^3x$ and equation (16) for the wave function, ψ which has been are normalized with respect to a finite spherical volume with inclusion of normalization factor N energy obtained as $E = \lambda E_p$ which means

$$E = -\sqrt{p^2 c^2 + m_o^2 c^4} \text{ and } E = +\sqrt{p^2 c^2 + m_o^2 c^4} \quad (18)$$

signifying the upper and lower energy continuum for $(\lambda = \pm 1)$ respectively

From the angle of the velocity of free particle, we consider the operator for the velocity in x-direction and compute it from the commutator with the Hamiltonian using the Heisenberg's representation in equation of motion where we have;

$$\dot{x} = \frac{i}{\hbar}\left[\hat{H},\mathrm{x}\right] = \frac{i}{\hbar}\left[c\left(\hat{\alpha}.p\right)+\hat{\beta}mc^2,x\right]$$
$$= \frac{i}{\hbar}\left[c\alpha_x p_x, x\right] = c\alpha_x \tag{19}$$

Similarly, we also

$$\dot{y} = c\alpha_y \text{ and } \dot{z} = c\dot{\alpha} \tag{20}$$

Or

$$v_i = c\alpha_i \tag{21}$$

$$v_i = c\alpha_x \tag{22}$$

Using Mathematica for the solution of the eigenvalue of the velocity for the x-direction, in equation (21), we have for

$$\lambda_1 = c$$

And

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\o\\0\\1\end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} \tag{23}$$

For $\lambda_2 = -c$

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} \tag{24}$$

C. Solution of Dirac Equation using Heisenberg's representation in equation of motion

To get a clear picture of this, we go further to look at the equation of motion in the Heisenberg's representation as well, where

$$\frac{d\hat{x}}{dt} = \frac{1}{i\hbar}\left[\hat{x},\hat{H}_f\right] = c\hat{\alpha} \tag{25}$$

$$\frac{d\hat{\alpha}}{dt} = \frac{1}{i\hbar}\left[\hat{\alpha},\hat{H}_f\right] = \frac{i}{\hbar}\left[\hat{\alpha},\hat{H}_f\right] - \frac{2i}{\hbar}\hat{\alpha}\hat{H}_f - \frac{2ic\hat{p}}{\hbar} - \frac{2i}{\hbar}\hat{\alpha}\hat{H}_f \tag{26}$$

As $\hat{p}$ and $\hat{H}_f$ are constant in time as $\left[\hat{p},\hat{H}\right] = 0 = \left[\hat{H}_f,\hat{H}_f\right]$, the former equation is integrated to give

$$\hat{\alpha}(t) = \left(\hat{\alpha}(\mathrm{o}) - \frac{c\hat{p}}{\hat{H}_f}\right)\exp\left(-2i\hat{H}_f/\hbar\right) + \frac{c\hat{p}}{\hat{H}_f} \tag{27}$$

Inserting equation (26) into equation (24), $\hat{x}(t)$ is explicitly expressed as

$$\hat{x}(t) = \left(\hat{\alpha}(o)\right) + \frac{c^2\hat{p}}{\hat{H}_f} + \left(\hat{\alpha}(\mathrm{o}) - \frac{c\hat{p}}{\hat{H}_f}\right)\frac{i\hbar c}{2\hat{H}_f}\exp(-2i\hat{H}_f/\hbar) \tag{28}$$

Now if this equation (27) is compared with classical equation of motion given by

$$\mathrm{x}_{cl}(\mathrm{t}) = \mathrm{x}_{cl}(0) + \left(\frac{c^2 p}{E_p}\right)_{cl} t \tag{29}$$

It will be observed that there is an extra term that is superposed with the uniform classical motion which is expressed as

$$\mathrm{x}_{cl}(t) = \left(\hat{\alpha}(\mathrm{o}) - \frac{c\hat{p}}{\hat{H}_f}\right)\frac{i\hbar c}{2\hat{H}_f}\exp(-2i\hat{H}_f/\hbar) \tag{30}$$

D. Classical relativistic velocity of free particle using Heisenberg Equation of Motion for acceleration

In this case also, we make use of the Heisenberg's equation of motion for the acceleration operator is given as

$$\dot{v}_i = \frac{i}{\hbar}\left[\hat{H}, v_i\right] = \frac{ic}{\hbar}\{\left(\hat{H}, \hat{\alpha}_i\right) - 2\hat{\alpha}_i\hat{H}\} \tag{31}$$

Now,

$$\left[\hat{H}, \hat{\alpha}_i\right] = \left[\sum_j c_j p_j \hat{\alpha}_j + \beta mc^2, \hat{\alpha}_i\right]_+ = \sum_j cp\{\hat{\alpha}_i\hat{\alpha}_j\} + mc^2\{\hat{\alpha}_i\beta\} \tag{32}$$

This eventually reduces to $2cp_i$

Since $\{\hat{\alpha}_i, \hat{\alpha}_j\} = \hat{\alpha}_i\hat{\alpha}_j + \hat{\alpha}_j\hat{\alpha}_i$ and $\{\hat{\alpha}_i, \beta\}$ =0,

We get

$$\dot{v}_i = \frac{ic}{\hbar}\left(2cp_i - 2\hat{\alpha}_i\hat{H}\right) = -\frac{ic}{\hbar}2\hat{H}\left(\hat{\alpha}_i - \frac{1}{\hat{H}}cp_i\right) \tag{33}$$

Now let $\eta_i = \hat{\alpha}_i - \frac{1}{\hat{H}}cp_i$

Since $v_i = c\hat{\alpha}_i$, then

$$\dot{v}_i = c\dot{\alpha}_i = c\dot{\eta}_i = \frac{ic}{\hbar}2\hat{H}\eta_i \tag{34}$$

Thus giving rise to first order differential equation in terms of η_i as

$$\dot{\eta}_i = -\frac{1}{\hbar}2\hat{H}\eta_i \tag{35}$$

The solution of this equation is

$$x_i(t) = x_i(o) + \frac{1}{\hat{H}}c^2 p_i t + \frac{c\hbar}{2i\hat{H}}\eta_i(o)\exp\left(-\frac{2i}{\hbar}\hat{H}t\right) \tag{36}$$

With $x_i(t)$ representing the position operator at the time t. while the position, x component of the velocity having two terms in which the second term signifies average motion of the particle while the first term is associated with a rapidly oscillatory motion that results from negative and positive Eigen energy state (16).

E. Solution from Semi-classical Projection Operator

From the angle of semi-classical point of view we involve particle of mass m and charge coupled to external time-independent electromagnetic field, [22] Dirac equation transformed in consideration to

$$E(\mathrm{x}) = \mathrm{grad}\,\phi(\mathrm{x}) \quad \text{and} \quad B(x) = rotA(x) \tag{37}$$

while the Hamiltonian now expressed as

$$\hat{H} = c\hat{\alpha}.\left(\frac{\hbar}{i}\nabla - \frac{e}{c}A(x)\right) + \beta mc^2 + e\phi(x) \tag{38}$$

Restriction attention to smooth potentials and then expression in equation (31) will considered to be self-ad joint within a give domain that will enable the definition of a unitary evolution

$$\hat{U}(t) := \exp-\frac{1}{\hbar}\hat{H}t \tag{39}$$

In order to decouple the particle and the anti-particle in the presence of the field, projection operators in phase space representation are constructed that would commute with the Hamiltonian and then choose the Wigner-Weyl calculus [23, 24]. With that, for each point (x, p) in the phase space in consideration of $4x4matrices$ we have two doubly degenerate eigenvalues

$$h_\pm(x,\mathrm{p}) = \mathrm{e}\,\phi(\mathrm{x}) \pm \sqrt{\left[\left(cp - eA(x)\right)^2 + m^2c^4\right]} \tag{40}$$

Which can be further solved (Emmrich and Weinsein, 1996) to arrive at

$$\varepsilon(x,\mathrm{p}) := \sqrt{\left[(cp - eA(x))^2 + m^2c^4\right]} \tag{41}$$

3. Result and Discussion

Various approach that have been employed to analyze the Dirac's equation yielded energy values consisting of two terms as presented in equation 12, here. It was observed from the solution that the energy value has both negative and positive state value respectively. In a similar view, another attempt was made by employing semi-classical operator and canonical formalism to solve for the energy eigen state in which energy momentum tensor was used in conjunction with the canonical energy momentum tensor, [18] there arose the same situation as in the first case whereby energy has two states associated to it, negative and positive states. From the analysis it was inferred there is interference between the negative energy state and positive state which results to the fluctuation that gives rise to the concept of zbw [12, 25]. According to Beuer and Horwitz, they stated that the presence of such state is needed to insure that charge conjugation has to be implemented in addition to time reversal in order to connect positive and negative time values, in agreement with Feynman's interpretation of the negative and positive energy states. [7, 21]

In order to explore further fact in support of the analysis, the results of eigenvalues of the velocity and acceleration in x-direction using Heisenberg representation as obtained using mathematic as in equations 22 and 33 it was found that the velocity has both negative and positive components. This is supported by the results from the theoretical solution as in equation 26 where it was seen that there was two terms in which the firs term represents a rapidly oscillatory motion is the zbw while the second term is associated with the average motion of the particle [16]. From the counterpart solution on using Heisenberg representation on equation of motion on acceleration, the result has two parts as well just like the former as in equation 36.

4. Summery

From the assessment, it has observed that whenever consideration of the free particle is based on the Dirac Hamiltonian, the resultant solution leads to eigenvalue spectrum of time operator that contains a gap between the positive and negative values which is similar to energy gap that appears in the energy spectrum that causes a jittery motion referred to as Zitterbewegung [25]. This is due to hypothical rapid oscillatory motion of relativistic elementary particle that obeys Dirac Equation which is as a result of interference between positive and negative energy state occasioned by what appeared to as fluctuation that occurs within the position of electron/ particle around the media.

Abbreviations

ZBW Zitterbewegung

Author Contributions

Emmanuel Ifeanyi Ugwu is the sole author. The author read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflicts of interest.

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